Transfer Matrix Representation of Flexible Airplanes in Gust Response Study

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The method of transfer matrices is used in the study of the response of flexible airplanes to random vertical gusts. Both nonhomogeneous and homogeneous gust fields can be treated within the present framework. A detailed formulation is carried out in the case of a slender wing with a nonuniform sweep angle, which is a possible configuration for large supersonic transports. Since the bending moment, torsional moment, and the shear are treated directly as unknowns rather than indirectly obtained from differentiating the deflections, this method is expected to yield a more accurate assessment of random stress levels.

Introduction

TWO types of formulations have been used in the determination of stress levels in flexible airplanes due to random gust excitations: the normal mode approach[1] and the influence coefficient approach.[2] In both of these approaches, the statistics of the deflections are first determined, and then the statistics of the moments and the shears required for structural design are obtained from successive differentiations (or finite-difference operations equivalent to differentiations). The accuracy of such results can be highly questionable. This paper is devoted to the development of a third method using the general ideas explored in a previous paper.[3] The new approach has the appeal that moments and shears can be treated as immediate unknowns along with the deflections; therefore, it is expected to yield more reliable results from a stress analyst's point of view. Furthermore, the coupling effect between connected parts of an airplane can be more directly accounted for, and damping and the unsteady aerodynamic forces can easily be included in the analysis.

Input-Output Relation

The term output refers to a random quantity of interest to the engineer, for example, the bending moment at a certain location on a structure. The random outputs result from a disturbance that is generally randomly distributed in space and randomly varying in time. However, a distributed disturbance can always be approximated by a number of concentrated ones. Referring to the concentrated disturbances as inputs, the problem is to determine the statistical information about a number of outputs from that of the inputs.

Assume that every input and every output possesses, in the sense of convergence in mean square, the following truncated Fourier transforms:

$$\bar{I}_j(\omega) = \frac{1}{2\pi}\int_{-T}^{T} I_j(t)e^{-i\omega t}dt$$
$$\bar{O}_k(\omega) = \frac{1}{2\pi}\int_{-T}^{T} O_k(t)e^{-i\omega t}dt \qquad (1)$$

For a stable linear system after the inputs have been applied for a sufficiently long time, there is the relation

$$\bar{O}_k = \sum_{j=1}^{n} H_{kj}(\omega)\bar{I}_j(\omega) \qquad (2)$$

where $H_{kj}(\omega)$ is the complex amplitude of the output at location k due to a sinusoidal input of a unit amplitude at location j. This function, $H_{kj}(\omega)$, is generally known as a frequency response function. Now if the $I_j(t)$ are jointly stationary, then the cross-spectral density of the jointly stationary outputs $O_j(t)$ and $O_k(t)$, defined as

$$\Phi_{O_jO_k}(\omega) = \lim_{T\to\infty} E\left[\frac{\pi\bar{O}_j(\omega)\bar{O}_k{}^*(\omega)}{T}\right] \qquad (3)$$

is given by

$$\Phi_{O_jO_k}(\omega) = \sum_{l=1}^{n}\sum_{m=1}^{n} H_{jl}(\omega)H_{km}{}^*(\omega)\Phi_{I_lI_m} \qquad (4)$$

where $E[\ ]$ denotes the mathematical expectation and an asterisk indicates the complex conjugate. In Eq. (4), $\Phi_{I_lI_m}$ is the cross-spectral density of $I_l(t)$ and $I_m(t)$, which is defined as in Eq. (3).

The cross-correlation function of two jointly stationary random processes is related to their cross-spectral density by the well-known Wiener-Khintchine relations

$$\phi_{O_jO_k}(\tau) = E[O_j(t)O_k(t+\tau)]$$
$$= \int_{-\infty}^{+\infty} \Phi_{O_jO_k}(\omega)e^{i\omega\tau}d\omega \qquad (5)$$

$$\Phi_{O_jO_k}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{O_jO_k}(\tau)e^{-i\omega\tau}d\tau \qquad (6)$$

When $j = k$ the cross correlation reduces to the autocorrelation and the cross-spectral density to the spectral density.

If the inputs $I_j(t)$ are nonstationary, then the outputs $O_k(t)$ are always nonstationary. For such cases Eq. (2) must be replaced by

$$\bar{O}_k(\omega) - E[\bar{O}_k(\omega)] = \sum_{j=1}^{n} H_{kj}(\omega)\{\bar{I}_j(\omega) - E[\bar{I}_j(\omega)]\} \qquad (2a)$$

Define a generalized cross-spectral density[4]:

$$\hat{\Phi}_{O_jO_k}(\omega_1, \omega_2) = \lim_{T=\infty} (E\{\bar{O}_j(\omega_1) - E[\bar{O}_j(\omega_1)]\}\{\bar{O}_k{}^*(\omega_2) - E[\bar{O}_k{}^*(\omega_2)]\}) \qquad (7)$$

when the right-hand side of Eq. (7) exists. It follows that

$$\hat{\Phi}_{O_jO_k}(\omega_1, \omega_2) = \sum_{l=1}^{n}\sum_{m=1}^{n} H_{jl}(\omega_1)H_{km}{}^*(\omega_2)\hat{\Phi}_{I_lI_m}(\omega_1, \omega_2) \qquad (8)$$

which has the same form as Eq. (4). The generalized cross-spectral density is related to the (nonstationary) covariance function as follows:

$$\hat{\Phi}_{O_jO_k}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{O_jO_k}(t_1, t_2)e^{i(\omega_1t_1-\omega_2t_2)}dt_1dt_2 \qquad (9)$$

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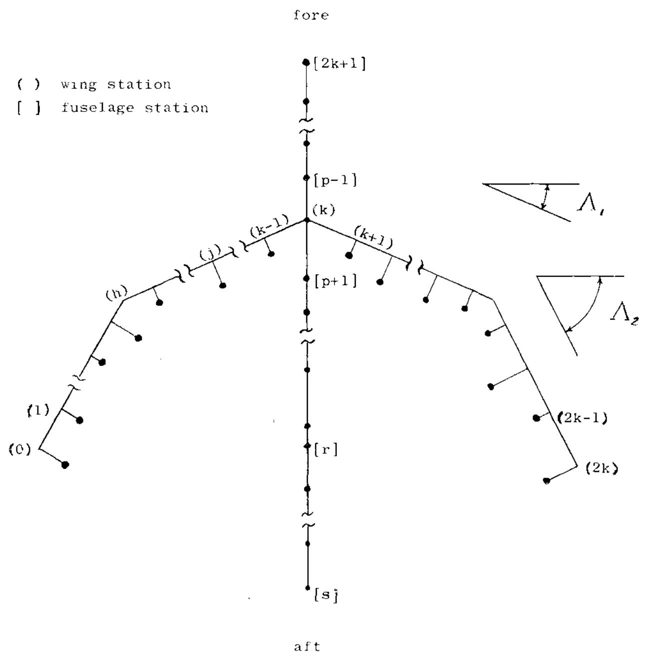


Fig. 1 Structural model for a large flexible airplane with variably swept wings.

and

$$K_{o_j o_k}(t_1, t_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\Phi}_{o_j o_k}(\omega_1, \omega_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2 \tag{10}$$

The nonstationary analysis is applicable to the transient regime as well, provided that the generalized cross-spectral densities involved exist.

In the present problem the disturbance is the gust up-wash when the airplane encounters atmospheric turbulence. Strictly speaking, the velocity of a gust is a random function of both time and space. In high-speed flights, however, it is generally permissible in the analysis of the airplane response to regard the gust velocity field to be random only in space. Thus the airplane is considered as flying through a random pattern of a gust which is "frozen" in space; and at a given location on the airplane it senses an input that is random in time. If the gust field is homogeneous, i.e., if the statistics of gust velocity are invariant with respect to a change of the origin of the reference coordinate axes, then the airplane senses stationary inputs; otherwise, it senses nonstationary inputs. The foregoing development, from Eqs. (1–10), provides a framework in which either a homogeneous or a nonhomogeneous gust can be treated.† Although this framework is applicable to a three-dimensional gust, for the sake of brevity, it will be assumed that the airplane response can be determined with sufficient accuracy by considering the effect of the vertical component of the gust alone.

The determination of the cross-spectral density for the inputs at two arbitrary locations, on an airplane in a homogeneous and in an isotropic‡ gust field, will be discussed in the Appendix.

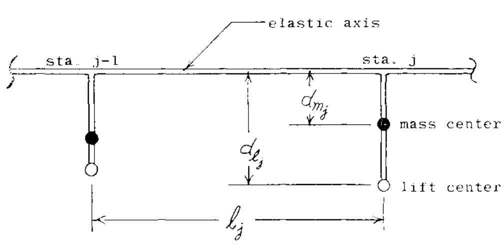
Structural Model of a Flexible Airplane

A large flexible airplane can be represented by a number of discrete masses connected by massless, elastic bars. Figure 1 shows such a representation suitable for the analysis of a high subsonic or low supersonic transport with a variably swept slender wing and a slender fuselage. For simplicity, it is assumed that the axes of the connected elastic bars and the mass centers lie in a horizontal plane. However, from the following formulation it will be clear that the analysis can easily be modified to account for dihedrals and/or a more general type of distribution of mass centers.

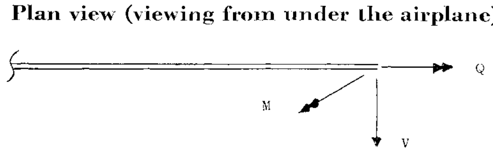
Consider a typical straight segment of an elastic bar representing one portion of the wing as shown in Fig. 2. The elastic bar is uniform§ from station to station as marked. Confining our attention to the airplane response to just the vertical component of gust, it is necessary to consider only the vertical deflection and the torsional deformation of the elastic bar. The forces in and the deformation of the elastic bar at a given location can be represented by a state vector $\mathbf{Z} = \{\phi, \delta, \theta, M, V, Q\}$. The components of this vector are, in the order as indicated, the angle of twist about the elastic axis, the vertical displacement, the slope in the vertical plane, the bending moment, the shear force, and the torsional moment. Given the length l_j, the bending rigidity $(EI)_j$, the torsional rigidity $(GJ)_j$, and the shear rigidity $(\eta GA)_j$ of this segment,¶ the state vector $\mathbf{Z}$ on the left of station j can be computed from that on the right of station j-1:

$$\mathbf{Z}_j{}^L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dfrac{l}{GJ} \\ 0 & 1 & l & -\dfrac{l^2}{2EI} & -\dfrac{l^3}{6EI} + \dfrac{l}{\eta GA} & 0 \\ 0 & 0 & 1 & -\dfrac{l}{EI} & -\dfrac{l^2}{2EI} & 0 \\ 0 & 0 & 0 & 1 & l & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_j \mathbf{Z}_{j-1}{}^R \tag{11}$$

The square matrix in Eq. (11) is known as a field transfer



a) Plan view (viewing from under the airplane)



b) Positive direction for bending moment M, shear V, and torsional moment Q

c) Positive direction for slope θ, vertical displacement δ, and angle of twist ϕ

Fig. 2 Structural model of a typical segment of wing.

† If the transient response due to a homogeneous gust is of interest, the nonstationary analysis can be used. To find the response at time t where $0 \leqslant t \leqslant t_a$, one assumes the inputs to be stationary in $[0, t_a]$ and equal to zero with probability one outside this time interval.

‡ Isotropicity implies that the statistics of the gust field are invariant with respect to a change of coordinate orientation.

§ A convenient but not an essential assumption.

¶ η is a coefficient depending on the shape of cross section, for example, $\eta = 1.2$ for a solid beam of a rectangular cross section.

matrix.**[5] The subscript j associated with this field matrix indicates that it is constructed using $l = l_j$, $EI = (EI)_j$, $GJ = (GJ)_j$, and $\eta GA = (\eta GA)_j$. Equation (11) is obtained for static deformations. In a dynamic problem when the motion is simple harmonic motion, Eq. (11) remains valid if each component in a static vector denotes a complex amplitude. In the following discussion, the dynamic case will be always implied.

The structural damping in an elastic segment can be accounted for by changing EI to $EI(1 + i\alpha)$, GJ to $GJ(1 + i\beta)$, and ηGa to $\eta GA(1 + i\beta)$ in the field matrix, where $i = (-1)^{1/2}$ and α and β are structural damping factors corresponding to normal and shear deformations, respectively. Write

$$\mathbf{Z}_j{}^L = \mathbf{F}_j\mathbf{Z}_{j-1}{}^R \tag{12}$$

then

$$\mathbf{F}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dfrac{l}{GJ(1+i\beta)} \\ 0 & 1 & l & \dfrac{-l^2}{2EI(1+i\alpha)} & \dfrac{-l^3}{6EI(1+i\alpha)} + \dfrac{l}{\eta GA(1+i\beta)} & 0 \\ 0 & 0 & 1 & \dfrac{-l}{EI(1+i\alpha)} & \dfrac{-l^2}{2EI(1+i\alpha)} & 0 \\ 0 & 0 & 0 & 1 & l & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_j \tag{13}$$

The relation between the state vectors $\mathbf{Z}_j{}^R$ and $\mathbf{Z}_j{}^L$ may be obtained from the equations of motion for station j. These equations involve an inertia force passing through the mass center, an inertia torque about an axis parallel to the elastic axis, and another inertia torque about an axis perpendicular to the elastic axis. For a simple harmonic motion with a frequency ω, the inertia force and torques are $m_j\ddot{\delta} = -\omega^2 m_j\delta_j$, $I_{pj}\ddot{\phi}_j = -\omega^2 I_{pj}\phi_j$, and $I_{qj}\ddot{\theta} = -\omega^2 I_{qj}\theta$. Furthermore, the continuities in the displacements ϕ, δ, and θ on both sides of station j must be maintained. It is convenient to combine the equations of motion and the continuity conditions in a matrix relation:

$$\mathbf{Z}_j{}^R = \mathbf{G}_j\mathbf{Z}_j{}^L + \mathbf{K}_j + \mathbf{L}_j \tag{14}$$

where $\mathbf{L}_j$ is associated with the lift resulting from the airplane penetrating a sinusoidal gust, $\mathbf{K}_j$ represents the contribution from the additional unsteady lift as a consequence of the deviation of the airplane motion from the steady flight, and

$$\mathbf{G}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_q\omega^2 & 1 & 0 & 0 \\ -md_m\omega^2 & -m\omega^2 & 0 & 0 & 1 & 0 \\ -(I_p + md_m{}^2)\omega^2 & -md_m{}^2\omega^2 & 0 & 0 & 0 & 1 \end{bmatrix}_j \tag{15}$$

The subscript j in Eq. (15) signifies that the mass m, the mass eccentricity d_m, and the mass polar moments of inertia I_p and I_q are those associated with the j station.

Equation (14) implies that the gust penetration effect $\mathbf{L}$ and the unsteady motion effect $\mathbf{K}$ are independent of each other. This is true when the unsteady motion of the airplane is small in comparison with the forward flight speed and the linearized aerodynamic theory is valid. Let W be the amplitude of the velocity of a vertical sinusoidal gust. The column $\mathbf{L}_j$ is dependent on W:

$$\mathbf{L}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -d_l \end{Bmatrix} Wg_j(\omega) \tag{16}$$

where $g_j(\omega)$ is a scalar function converting the (complex) amplitude of the gust velocity to the (complex) amplitude of the penetration lift. Here, the penetration lift is considered as a concentrated sinusoidally varying force, which is transmitted through a rigid arm to station j. [Thus $g_j(\omega)$ is proportional to the contributing area assigned to this station.] In the special case of an infinite long straight wing in an incompressible flow, the $g(\omega)$ function is proportional to the well-known Sears function, which is often expressed in terms of the reduced frequency $k = \omega b/U$, where b is the chord width of the wing and U is the forward flight speed.[6, 7] In general, this g function is a function of the frequency ω, the sweep angle Λ, and (at least) the local geometry of the lifting surface. The exact expression of this function in the most general case remains unknown; in some cases it may be approximated by modifying the Sears function.

The column $\mathbf{K}_j$ may be written as follows:

$$\mathbf{K}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \phi f_2(\omega) + \delta f_1(\omega) \\ \phi f_3(\omega) + \delta f_4(\omega) \end{Bmatrix}_j \tag{17}$$

The physical meanings of the functions $f_1(\omega)$ through $f_4(\omega)$ are clear. For example, $f_1(\omega)$ is the complex amplitude of the unsteady lift due to a unit sinusoidal vertical translation at the station considered, and $f_2(\omega)$ is the complex amplitude of the unsteady lift due to a unit sinusoidal angle of twist, etc. From the reciprocal relation of a linearized aerodynamic theory it may be concluded that $f_2(\omega) = f_4(\omega)$. For a rigid wing with an infinite span the f functions involve the Theordosen functions. In the general case, however, these f functions are not available in the literature although it is clear that they must be functions of the frequency, the sweep angle, and (at least) the local geometry of the lifting surface; and they sometimes may be approximated by modifying the Theordosen functions.

Since the column $\mathbf{K}_j$ depends on the displacements δ and ϕ, it is convenient to rewrite Eq. (14) in the following way:

$$\mathbf{Z}_j{}^R = \mathbf{P}_j\mathbf{Z}_j{}^L + \mathbf{L}_j \tag{18}$$

where

$$\mathbf{P}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_q\omega^2 & 1 & 0 & 0 \\ -md_m\omega^2 + f_2(\omega) & -m\omega^2 + f_1(\omega) & 0 & 0 & 1 & 0 \\ -(I_p + md_m{}^2)\omega^2 + f_3(\omega) & -md_m\omega^2 + f_2(\omega) & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

** A more complicated field matrix results if the bending and shear rigidities in the segment are nonuniform.

This square matrix is called a point transfer matrix[5]; it relates the state vectors on two sides of a station if the external lift is absent.

It is interesting to note that both the field transfer matrix $\mathbf{F}$ and the point transfer matrix $\mathbf{P}$ are symmetrical about the cross diagonal. This is always true for a linear problem such as the present one when the order of the components (with proper signs) in the related state vectors is properly arranged.†† A physical insight is that the same relation exists either transferring the state from left to right or from right to left, either across a field or a point.

A different situation arises when the transfer of state involves a change in the coordinate system. An example is found at station h in Fig. 1 where the positive directions for Q and M, and for ϕ and θ, are rotated by an angle $\gamma = \Lambda_2 - \Lambda_1$. For simplicity, assuming that no mass is attached nor lift is fed in at this station, the change can be accomplished by introducing a point transfer matrix

$$\mathbf{P}_h = \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin\gamma & 0 & \cos\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma & 0 & -\sin\gamma \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\gamma & 0 & \cos\gamma \end{bmatrix} \tag{20}$$

This transfer matrix is again symmetrical about the cross diagonal.

A more complicated situation exists at the station where the wing joins with the fuselage. Referring to Fig. 1, this station is k on the wings, but it is also p on the fuselage. Consider the transfer of state along the fuselage proceeding in the fore-to-aft direction:

$$\mathbf{Z}_p{}^A = \mathbf{P}_p\mathbf{Z}_p{}^F + \mathbf{L}_p + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{Bmatrix} \tag{21}$$

Equation (21) is essentially the same as Eq. (18) except for the last column, which accounts for the bending moment X, the vertical shear Y, and the torsional moment Z transmitted from the wing. Here X, Y, Z are expressed according to the coordinate system of the fuselage. Since X, Y, Z must also be the reactions from the fuselage acting at station k of the wing, there is the relation

$$\mathbf{Z}_k{}^R = \mathbf{P}_k\mathbf{Z}_k{}^A - \left[\begin{array}{c|ccc} \mathbf{O} & & \mathbf{O} & \\ \hline & \sin\Lambda_1 & 0 & \cos\Lambda_1 \\ \mathbf{O} & 0 & 1 & \\ & -\cos\Lambda_1 & 0 & \sin\Lambda_1 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{Bmatrix} \tag{22}$$

where $\mathbf{P}_k$ may be obtained from Eq. (20) by replacing γ by $2\Lambda_1$. The quantities X, Y, Z may be eliminated since they appear both in Eqs. (21) and (22). Furthermore, compatibility in displacements requires

$$\begin{Bmatrix} \phi \\ \delta \\ \theta \end{Bmatrix}_k^L = \begin{bmatrix} -\sin\Lambda_1 & 0 & \cos\Lambda_1 \\ 0 & 1 & 0 \\ -\cos\Lambda_1 & 0 & -\sin\Lambda_1 \end{bmatrix} \begin{Bmatrix} \phi \\ \delta \\ \theta \end{Bmatrix}_p^F \tag{23}$$

The foregoing transfer matrices provide a means by which the state vector at any point on the system of the elastic axes may be expressed in terms of that at another point. With the boundary conditions and the external excitations specified, the state vector at any point can then be obtained from a straightforward matrix algebra. Some details will be given in the following determination of frequency response functions.

†† This is precisely the reason for choosing the order ϕ, δ, θ, M, V, Q, and the opposite signs for θ and M. See Fig. 2.

Determination of Frequency Response Functions

The frequency response function $H_{rj}(\omega)$ is obtained by assuming that only station j is excited by a sinusoidal gust up-wash of a unit amplitude (i.e., $W = 1$). The response of interest may be either ϕ, δ, θ, M, V, or Q at a point r. To illustrate, let j be a station on the left side of the wing and r be immediately aft of station r on the rear fuselage as shown in Fig. 1. Denote by ${}_l\mathbf{T}_m$ and ${}_l\mathbf{U}_m$ the 6×6 matrices obtained from the chain rules:

$$\begin{aligned} {}_l\mathbf{T}_m &= \mathbf{F}_l\mathbf{P}_{l-1}\mathbf{F}_{l-1}\ldots\mathbf{L}_{m+1}\mathbf{P}_m \\ {}_l\mathbf{U}_m &= \mathbf{P}_l\mathbf{F}_l\mathbf{P}_{l-1}\mathbf{F}_{l-1}\ldots\mathbf{L}_{m+1}\mathbf{P}_m = \mathbf{P}_l\,{}_l\mathbf{T}_m \end{aligned} \tag{24}$$

With only a unit sinusoidal input at station j, there is the relation,

$$\begin{Bmatrix} \phi \\ \delta \\ \theta \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{2k}^R = {}_{2k}\mathbf{U}_0 \begin{Bmatrix} \phi \\ \delta \\ \theta \\ 0 \\ 0 \\ 0 \end{Bmatrix}_0^L + {}_{2k}\mathbf{U}_j \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -d_l \end{Bmatrix} g_j(\omega) - {}_{2k}\mathbf{U}_k \left[\begin{array}{c|ccc} \mathbf{O} & & \mathbf{O} & \\ \hline & \sin\Lambda_1 & 0 & \cos\Lambda_1 \\ \mathbf{O} & 0 & 1 & 0 \\ & -\cos\Lambda_1 & 0 & \sin\Lambda_1 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{Bmatrix} \tag{25}$$

where the boundary conditions $\{M, V, Q\}_0 = \{M, V, Q\}_{2k}{}^R = \{0, 0, 0\}$ have been inserted in this equation. Consider the change of states along the fuselage:

$$\begin{Bmatrix} \phi \\ \delta \\ \theta \\ 0 \\ 0 \\ 0 \end{Bmatrix}_s^A = {}_s\mathbf{U}_{2k+1} \begin{Bmatrix} \phi \\ \delta \\ \theta \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{2k+1}^F + {}_s\mathbf{U}_p \begin{Bmatrix} 0 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{Bmatrix} \tag{26}$$

Now,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = {}_{2k}\begin{bmatrix} u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \\ u_{61} & u_{62} & u_{63} \end{bmatrix}_0 \begin{Bmatrix} \phi \\ \delta \\ \theta \end{Bmatrix}_0^L + \begin{bmatrix} u_{45} & u_{46} \\ u_{55} & u_{56} \\ u_{65} & u_{66} \end{bmatrix} \begin{Bmatrix} -1 \\ -d_l \end{Bmatrix}_j g_j(\omega) - {}_{2k}\begin{bmatrix} u_{44} & u_{45} & u_{46} \\ u_{54} & u_{55} & u_{56} \\ u_{64} & u_{65} & u_{66} \end{bmatrix}_k \begin{bmatrix} \sin\Lambda_1 & 0 & \cos\Lambda_1 \\ 0 & 1 & 0 \\ -\cos\Lambda_1 & 0 & \sin\Lambda_1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{27}$$

can be extracted from Eq. (25), where each u_{lm} represents the (l, m) element taken from the indicated U matrix (the indices constructed according to the chain rule). Similarly from Eq. (26),

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = {}_s\begin{bmatrix} u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \\ u_{61} & u_{62} & u_{63} \end{bmatrix}_{2k+1} \begin{Bmatrix} \phi \\ \delta \\ \theta \end{Bmatrix}_{2k+1}^F + {}_s\begin{bmatrix} u_{44} & u_{45} & u_{46} \\ u_{54} & u_{55} & u_{56} \\ u_{44} & u_{65} & u_{66} \end{bmatrix}_p \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{28}$$

Solving for $\{X, Y, Z\}$,

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = -\begin{bmatrix} u_{44} & u_{45} & u_{46} \\ u_{54} & u_{55} & u_{56} \\ u_{64} & u_{65} & u_{66} \end{bmatrix}_p^{-1} {}_s\begin{bmatrix} u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \\ u_{61} & u_{62} & u_{63} \end{bmatrix}_{2k+1} \begin{Bmatrix} \phi \\ \delta \\ \theta \end{Bmatrix}_{2k+1}^F \tag{29}$$

Substituting into Eq. (27) gives

$$\begin{Bmatrix}0\\0\\0\end{Bmatrix} = {}_{2k}\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_0 \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_0 + {}_{2k}\begin{bmatrix}u_{45} & u_{46}\\ u_{55} & u_{56}\\ u_{65} & u_{66}\end{bmatrix}_j \begin{Bmatrix}-1\\ -d_l\end{Bmatrix}{}_j g_j(\omega) +$$
$${}_{2k}\begin{bmatrix}u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_k \begin{bmatrix}\sin\Lambda_1 & 0 & \cos\Lambda_1\\ 0 & 1 & 0\\ -\cos\Lambda_1 & 0 & \sin\Lambda_1\end{bmatrix} {}_s\begin{bmatrix}u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_p^{-1} \times$$
$${}_s\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_{2k+1} \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_{2k+1} \quad (30)$$

Obviously the vectors $\{\phi, \delta, \theta\}_0{}^L$ and $\{\phi, \delta, \theta\}_{2k+1}{}^F$ are not independent, and their relation may be found as follows: Write

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\\ M\\ V\\ Q\end{Bmatrix}_p^F = {}_p\mathbf{T}_{2k+1} \begin{Bmatrix}\phi\\ \delta\\ \theta\\ 0\\ 0\\ 0\end{Bmatrix}_{2k+1}^F \quad (31)$$

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\\ M\\ V\\ Q\end{Bmatrix}_k^L = {}_k\mathbf{T}_0 \begin{Bmatrix}\phi\\ \delta\\ \theta\\ 0\\ 0\\ 0\end{Bmatrix}^L + {}_k\mathbf{T}_j \begin{Bmatrix}0\\ 0\\ 0\\ 0\\ -1\\ -d_l\end{Bmatrix} g_j(\omega) \quad (32)$$

These equations contain

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_p^F = {}_p\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{32}\end{bmatrix}_{2k+1} \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_{2k+1}^F \quad (33)$$

and

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_k^L = {}_k\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_0 \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_0^L + {}_k\begin{bmatrix}t_{15} & t_{16}\\ t_{25} & t_{26}\\ t_{35} & t_{36}\end{bmatrix}_j \begin{Bmatrix}-1\\ -d_l\end{Bmatrix} g_j(\omega) \quad (34)$$

Thus

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_0^L = -{}_k\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_0^{-1} {}_k\begin{bmatrix}t_{15} & t_{16}\\ t_{25} & t_{26}\\ t_{35} & t_{36}\end{bmatrix}_j \begin{Bmatrix}-1\\ -d_l\end{Bmatrix} g_j(\omega) +$$
$${}_k\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_0^{-1} \begin{bmatrix}-\sin\Lambda_1 & 0 & \cos\Lambda_1\\ 0 & 1 & 0\\ -\cos\Lambda_1 & 0 & -\sin\Lambda_1\end{bmatrix} \times$$
$${}_p\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_{2k+1} \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_{2k+1}^F \quad (35)$$

where the relation (23) has been used. Let

$$\mathbf{B} = {}_{2k}\begin{bmatrix}u_{45} & u_{46}\\ u_{55} & u_{56}\\ u_{65} & u_{66}\end{bmatrix}_j - {}_{2k}\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_0 \times$$
$${}_k\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_0^{-1} {}_k\begin{bmatrix}t_{15} & t_{16}\\ t_{25} & t_{26}\\ t_{35} & t_{36}\end{bmatrix}_j \quad (36)$$

and

$$\mathbf{C} = {}_{2k}\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_0 {}_k\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_0^{-1} \times$$
$$\begin{bmatrix}-\sin\Lambda_1 & 0 & \cos\Lambda_1\\ 0 & 1 & 0\\ -\cos\Lambda_1 & 0 & -\sin\Lambda_1\end{bmatrix} {}_p\begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}_{2k+1} +$$
$${}_{2k}\begin{bmatrix}u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_k \begin{bmatrix}\sin\Lambda_1 & 0 & \cos\Lambda_1\\ 0 & 1 & 0\\ -\cos\Lambda_1 & 0 & \sin\Lambda_1\end{bmatrix} \times$$
$${}_s\begin{bmatrix}u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_p^{-1} {}_s\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_{2k+1} \quad (37)$$

Eq. (30) reduces to

$$\begin{Bmatrix}0\\0\\0\end{Bmatrix} = \mathbf{B}\begin{Bmatrix}-1\\ -d_l\end{Bmatrix}{}_j g_j(\omega) + \mathbf{C}\begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_{2k+1}^F \quad (38)$$

Rearranging

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix} = \mathbf{C}^{-1}\mathbf{B}\begin{Bmatrix}-1\\ -d_l\end{Bmatrix} g_j(\omega) \quad (39)$$

Since

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\\ M\\ V\\ Q\end{Bmatrix}_r^A = {}_r\mathbf{U}_{2k+1}\begin{Bmatrix}\phi\\ \delta\\ \theta\\ 0\\ 0\\ \theta\end{Bmatrix}_{2k+1}^F + {}_r\mathbf{U}_p\begin{Bmatrix}0\\ 0\\ 0\\ X\\ Y\\ Z\end{Bmatrix} =$$
$${}_r\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ u_{21} & u_{22} & u_{23}\\ u_{31} & u_{32} & u_{33}\\ u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_{2k+1} \begin{Bmatrix}\phi\\ \delta\\ \theta\end{Bmatrix}_{2k+1}^F + {}_r\begin{bmatrix}u_{14} & u_{15} & u_{16}\\ u_{24} & u_{25} & u_{26}\\ u_{34} & u_{35} & u_{36}\\ u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_p \begin{Bmatrix}X\\ Y\\ Z\end{Bmatrix} \quad (40)$$

It follows from Eqs. (29) and (39) that

$$\begin{Bmatrix}\phi\\ \delta\\ \theta\\ M\\ V\\ Q\end{Bmatrix}_r^A = \left({}_r\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ u_{21} & u_{22} & u_{23}\\ u_{31} & u_{32} & u_{33}\\ u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_{2k+1} - {}_r\begin{bmatrix}u_{14} & u_{15} & u_{16}\\ u_{24} & u_{25} & u_{26}\\ u_{34} & u_{35} & u_{36}\\ u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_p \times \right.$$
$$\left. {}_s\begin{bmatrix}u_{44} & u_{45} & u_{46}\\ u_{54} & u_{55} & u_{56}\\ u_{64} & u_{65} & u_{66}\end{bmatrix}_p^{-1} {}_s\begin{bmatrix}u_{41} & u_{42} & u_{43}\\ u_{51} & u_{52} & u_{53}\\ u_{61} & u_{62} & u_{63}\end{bmatrix}_{2k+1} \right) \mathbf{C}^{-1}\mathbf{B}\begin{Bmatrix}-1\\ -d_l\end{Bmatrix} g_j(\omega) \quad (41)$$

This equation gives the angle of twist, deflection, slope, moment, shear, and torsional moment immediately aft of station r (on the rear fuselage) resulting from a unit sinusoidal gust up-wash at station j (on the left side of the wing). By definition, these are the H_{rj} functions, each of which corresponds to one type of response being considered. The frequency response functions corresponding to other pairs of stations can be found in an analogous manner.

Once the frequency response functions are known, the cross-power spectrum of the response at any two stations is readily obtained from Eq. (4) if the gust field is homogeneous or from Eq. (8) if it is nonhomogeneous.

Concluding Remarks

The method outlined in this paper permits the separate determination of six types of response, three of which are the displacements (ϕ, δ, θ) and the other three are the force and moments (M, V, Q). The displacement types of response are of immediate interest to flight stability and the passenger's comfort, whereas the force and moments are needed in computing the stresses.

In the application of the present method, the values (both real and imaginary parts) of the frequency response functions are computed at each given frequency. The process is a straightforward matrix manipulation and the knowledge of the natural frequencies and the mode shapes are not needed to carry it through. The transfer matrix formulation can also be used for the determination of natural frequencies and the normal-modes of the entire flexible airplane if so desired, but the computation would generally require an iterative procedure. The author's experience in another problem[8] has shown that convergence in the iteration process for mode shapes can be very slow. The fact that the determination of frequency response functions requires no iteration suggests that the present method is more suitable for use in the study of the steady-state forced vibrations or forced random vibrations than for free vibrations.

Appendix: Cross-Spectral Density of Inputs, Homogeneous Isotropic Gust

Let 0 xyz be the reference coordinates attached to the moving airplane, and let x - z plane be the plane of symmetry. The airplane is assumed to be moving at a constant velocity u in the negative direction of the x axis. The cross-correlation function for the inputs at locations j and l on the airplane is then

$$\phi_{IjIl}(\tau) = E[W(x_j - ut_1, y_j)\cdot W(x_l - ut_2, y_l)] \qquad \text{(A1)}$$

where W is the random gust vertical velocity. In the case of a homogeneous gust, the right-hand side of Eq. (A1) may be replaced by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{ww}(k_1, k_2)\cdot \exp(i\{k_1[(x_j - ut_1) - (x_l - ut_2)] + k_2(y_j - y_l)\})\cdot dk_1 dk_2 \qquad \text{(A2)}$$

Here $\Phi_{ww}(k_1, k_2)$ is the two-dimensional mean square spectral density of the gust vertical velocity in the domain of wave numbers k_1 and k_2. Further restricting the gust field to be isotropic,[9]

$$\Phi_{ww}(k_1, k_2) = \frac{3}{4}\frac{\sigma^2 L^2}{\pi}\frac{L^2(k_1^2 + k_2^2)}{[1 + L^2(k_1^2 + k_2^2)]^{5/2}} \qquad \text{(A3)}$$

In Eq. (A3), L is the scale of turbulence and σ^2 is the variance of the gust vertical velocity. Denoting $\xi = x_j = x_l$, $\eta = y_j - y_l$, and $\tau = t_1 - t_2$, Eqs. (A1–A3) may be combined into

$$\phi_{IjIl}(\tau) = \frac{3}{4}\frac{\sigma^2 L^2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{L^2(k_1^2 + k_2^2)}{[1 + L^2(k_1^2 + k_2^2)]^{5/2}}\cdot \exp[i(k_1\xi - k_1 u\tau + k_2\eta)]dk_1 dk_2 \qquad \text{(A4)}$$

The cross-spectral density of the inputs is the Fourier transform of (A4)

$$\Phi_{IjIl}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi_{IjIl}(\tau)e^{-i\omega\tau}d\tau \qquad \text{(A5)}$$

Substituting Eq. (A4) into Eq. (A5), and interchanging the order of integration,

$$\Phi_{IjIl}(\omega) = \frac{3}{4}\frac{\sigma^2 L^2}{\pi}\exp\left(-i\frac{\omega}{u}\xi\right)\cdot \int_{-\infty}^{\infty}\frac{L^2[(\omega/u)^2 + k_2^2]}{\{1 + L^2[(\omega/u)^2 + k_2^2]\}^{5/2}}e^{ik_2\eta}\,dk_2 \qquad \text{(A6)}$$

where use has been made of the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k_1 u + \omega)\tau}d\tau = \delta(k_1 u + \omega) \qquad \text{(A7)}$$

The integral in Eq. (A6) can be evaluated with the aid of existing tables[10]: the result is

$$\Phi_{IjIl}(\omega) = \frac{2\sigma^2 L}{\pi}\exp\left(-i\frac{\omega}{u}\xi\right)\cdot \left\{\left(\frac{\omega}{u}\right)^2\left[\frac{\eta^2}{4a^2}K_2\left(a\frac{\eta}{L}\right)\right] + \left(\frac{1}{2a}\right)^2\frac{\alpha\eta}{L}\left[3K_1\left(\frac{a\eta}{L}\right) - \frac{a\eta}{L}K_2\left(\frac{a\eta}{L}\right)\right\}\right] \qquad \text{(A8)}$$

where K_n are the modified Bessel functions of the second kind, and $a = [1 + (L\omega/u)^2]^{1/2}$.

The spectral density of the input (at one single location) is obtained by letting $\xi = \eta = 0$ in Eq. (A8). Observing that as $z \to 0$,

$$zK_1(z) \to 1 \qquad z^2K_2(z) \to 2 \qquad \text{(A9)}$$

it follows that

$$\Phi_{IjIl}(\omega) = \frac{\sigma^2 L}{2\pi}\frac{1 + 3(\omega L/u)^2}{[1 + (\omega L/u)^2]^2} \qquad \text{(A10)}$$

Equation (A10) is a well-known result frequently used in the study of airplane response to a one-dimensional random gust.[11]

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